

A topological origin for Dark Energy

J. Lorca Espiro^{a,b}, M Le Delliou^c

^a*Departamento de Ciencias Físicas, Facultad de Ingeniería, Ciencias y Administración, Universidad de La Frontera, Avda. Francisco Salazar 01145, Casilla 54-D Temuco, Chile.*

^b*Departamento de Física Matemática do Instituto de Física Universidade de São Paulo CP 66.318 05314-970, São Paulo, SP Brasil.*

^c*Instituto de Física Teórica, Universidade Estadual de São Paulo (IFT-UNESP), Rua Dr. Bento Teobaldo Ferraz 271, Bloco 2 - Barra Funda, 01140-070 São Paulo, SP Brazil*

Abstract

Cosmology struggles with the theoretical problems generated by the observed value and recent emergence of a cosmological constant, in the standard model of cosmology, i.e. the concordance model. We propose to provide a more natural explanation for its value than the conventional quantum vacuum energy in the guise of topological invariants. Introducing topological classes densities as Lagrange multipliers, an effective cosmological constant is generated. General Relativity is reestablished by cancelling the torsion thus generated, which provides constraints on the invariants and yield the form of the effective cosmological constant. As it is divided by the total volume of spacetime, its small value compared to the Planck scale is therefore natural. It also provides a direct measurement of the global Euler number.

Keywords: general relativity, differential geometry, topological invariants, torsion two-form, space-time topology, cosmological constant, cosmology

1. Introduction

Einstein-Cartan theories generalise the curved spacetime approach of general relativity (GR) to allow for the presence of torsion of the manifold [1, 2, 3, 4]. The extreme case of curvatureless theory with torsion can yield the Teleparallel Equivalent to GR [5, 6]. Further freedom can be gained by

Email addresses: javier.lorca.espiro@ifusp.br (J. Lorca Espiro), delliou@ift.unesp.br (M Le Delliou)

modifying the type of manifold topology [7, 8] and boundary terms [9] of the Gravitational action. In a previous paper [10], the classical Einstein-Hilbert action boundary conditions were modified by coupling characteristic topological classes densities as Lagrange multipliers, which lead to the appearance of a term that can be interpreted as an *effective cosmological constant*.

In cosmology, the need for such a constant to describe the observed universe arose after the detection of the stronger than expected fainting of standard candles provided by type Ia supernovae [11], and was later confirmed by observations of the cosmic microwave background radiation, combined with clusters and baryon acoustic oscillation measurements [12, 13, 14, 15]. However the common interpretation of such constant as a quantum vacuum energy collides with the theoretical conundrums caused by the discrepancy between observed value of Λ and evaluations of the quantum vacuum energy, i.e. of order the Planck scale. This gives rise to the coincidence problem [16, 17, 18, 19] and to the fine tuning problem [20, 21]. The coincidence problem concerns the fact that the value of the cosmological constant seems chosen such that its dominance in the cosmic energy density balance arose very close to the present epoch. As it is a constant the fine tuning problem, which is more directly concerned with the discrepancy between its value and the Planck scale which ruled the quantum vacuum scale at early time and should have given initial values for all running constants, is embedded together with the coincidence problem in the value of the cosmological constant.

In this paper, we consider the possibility that such a constant is not generated from quantum vacuum considerations at the formation of the universe but rather has a topological origin. Contrary to [10], where the solutions that correspond to GR settings don't depend on the topology, the effective cosmological constant in the solutions we are investigating here are completely topological, that is we have no cosmological constant a priori, assuming some symmetry consideration will have set the bare constant to zero. Starting from the inclusion in the GR action of the topological classes densities consistent with a space-time manifold, i. e. the Nieh-Yan, the Euler and the Pontryagin classes, we introduce the torsion, for which the restriction to GR will lead to some more constraints. In this context, a topological effective cosmological constant naturally arises. Due to its nature, it renders moot the fine tuning problem since its scale is no longer connected to the large Planck density, but rather to the small inverse manifold volume, and we propose to link its value with the topology of the universe.

The paper organises as follows: we will first introduce the articulation of the Einstein-Hilbert action with topological invariants in Sec. 2. We will then return to the action and present the supplementary dynamics introduced by these invariants in Sec. 3. Then, Sec. 4 will extract the effective cosmological constant from imposing that gravity shall be described by GR. Finally, we will discuss our findings and conclude in Sec. 5.

2. General Relativity and topological invariants

In this section we will describe how introducing topological invariants in the action of GR generates the emergence of torsion.

2.1. General relativity and Cartan formalism

The standard General Relativity (GR) can be obtained by considering the following action over a four manifold \mathcal{M}

$$S_{EH}[e^c, \bar{\omega}^a{}_b] = \int_{\mathcal{M}} \frac{1}{\kappa} \left(\epsilon^{abcd} e_a \wedge e_b \wedge \bar{R}_{cd} + \frac{\Lambda}{6} \epsilon \right), \quad (1)$$

where Λ is the cosmological constant, e^a denotes the 1-form frame fields or vierbein, κ is the Gravitation constant, ϵ is the volume form $\epsilon \doteq \frac{1}{4!} \epsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d$ and $\bar{\omega}^a{}_b$ is the Levi-Civita connection 1-form. The later is equivalent to the Einstein-Hilbert action, where \bar{R}_{ab} is the curvature two-form and is defined by

$$\bar{R}^{ab} \doteq d\bar{\omega}^{ab} + \bar{\omega}^a{}_c \wedge \bar{\omega}^{cb}. \quad (2)$$

where $d(\cdot)$ is the exterior derivative. The Palatini variation of (1) gives the Einstein's Field equations and the null torsion condition

$$\bar{T}^a \doteq d_{\bar{\omega}} e^a = 0.$$

where $d_{\bar{\omega}}(\cdot)$ denotes the covariant derivative with respect to the connection $\bar{\omega}^{ab}$. Due to the nullity of the previous equation, it is possible to write the action as well as the connection entirely in terms of the vierbein.

2.2. Adding characteristic classes to GR: The appearance of Torsion

On an oriented space-time \mathcal{M} the characteristic classes of the tangent bundle available and consistent with the space-time structure are the Euler class $e(\mathcal{M})$, the Pontryagin class $p_1(T\mathcal{M}) \in H^4(\mathcal{M}; \mathbb{Z}) = \mathbb{Z}$ and the Chern

class $c_2(\mathcal{M}) = c_2(T\mathcal{M}) \in H^4(\mathcal{M}, \mathbb{Z})$ (Nieh-Yan) [22, 7, 8, 23]. The previous have the following representations as 4-forms, respectively

$$C_P = R^a_b \wedge R^b_a, \quad (3)$$

$$C_E = \epsilon_{abcd} R^{ab} \wedge R^{cd}, \quad (4)$$

$$C_{NY} = T^a \wedge T_a - R_{ab} \wedge e^a \wedge e^b. \quad (5)$$

As it is clear from the previous equations, the addition of characteristic classes directly in the Lagrangian will result in a non zero torsion T^a . A way to acknowledge these new settings is through what is known as *first order formalism* [24, 25, 26, 27, 28, 29, 30], where the key is to consider a generalized connection one-form to be

$$\omega^{ab} \doteq \bar{\omega}^{ab} + K^{ab}, \quad (6)$$

$$T^a \doteq d_\omega e^a, \quad (7)$$

where K^{ab} is called the *contortion* one form. It is immediate to note that the previous equation leads immediately to the relation

$$T^a = K^{ab} \wedge e_b. \quad (8)$$

Is in this sense that the contortion K^{ab} is the responsible for the appearance of torsion T^a . Similarly, the curvature two-form can thus be written in the following way

$$R^{ab} = \bar{R}^{ab} + d_\omega K^{ab} - K^a_c \wedge K^{cb}, \quad (9)$$

where equations (2) and (6) have been used. To complete the picture, the Bianchi identities

$$d_\omega T^a = R^a_b \wedge e^b, \quad (10)$$

$$d_\omega R^a_b = 0. \quad (11)$$

hold for R_{ab} and T_a .

3. Einstein-Hilbert with topological invariants

As it was done in [10], when considering the following family of actions

$$\begin{aligned} S[e^c, \omega^a_b, \varphi_i, n_i] &\doteq S_{EH}[e^c, \omega^a_b] - \int_{\mathcal{M}} \sum_i \varphi_i \left(C_i - \frac{n_i}{V_{\mathcal{M}}} \epsilon \right) \\ &= S_{EH}^{eff}[e^c, \omega^a_b, \varphi_i, n_i] - \int_{\mathcal{M}} \sum_i \varphi_i C_i, \end{aligned} \quad (12)$$

where $i = P, NY, E$, and $V_{\mathcal{M}} = \int_{\mathcal{M}} \epsilon$ the additional parameters n_i are the Nieh-Yan, Pontryagin and the Euler numbers, respectively, $\varphi_i : \mathcal{M} \rightarrow \mathbb{R}$ are coupling zero forms and C_i are the characteristic classes defined in (3-5), where we have written

$$S_{EH}^{eff}[e^c, \omega^a_b, \varphi_i, n_i] \doteq \int_{\mathcal{M}} \frac{1}{\kappa} \left(\epsilon^{abcd} e_a \wedge e_b \wedge R_{cd} + \frac{\Lambda_{eff}}{6} \epsilon^{abcd} e_a \wedge e_b \wedge e_c \wedge e_d \right), \quad (13)$$

an apparent quasi topological *effective cosmological constant* seems to arise, which has been defined as

$$\Lambda_{eff}[\Lambda, \varphi_i, n_i] \doteq \Lambda + \frac{1}{4V_{\mathcal{M}}} \sum_i n_i \varphi_i. \quad (14)$$

Note that the first form of equation (12) shows explicitly how the Lagrange multipliers are inserted containing only *non-holonomic* restrictions.

Regarding the n_i parameters, they have been inserted joined with the characteristic densities in a way that will force them to be integers by construction. We will elaborate further on this aspect, let us recall that for *any simply connected oriented smooth 4-manifold* it has been proven that the topological numbers $n_i \in \mathbb{Z}$ [31, 22]. That the manifold is oriented comes from the fact that when this is so it is necessary to have a differentiable four-form which in this case is ensured by the existence of the volume form $\epsilon = \frac{1}{4!} \epsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d$ inserted directly in Eq. (12). On the other hand, the couplings φ_i fulfill

$$\begin{aligned} C_i &= \frac{n_i}{V_{\mathcal{M}}} \epsilon, \\ &= \frac{n_i}{V_{\mathcal{M}}} \star (1), \\ &= \left\langle \frac{n_i}{V_{\mathcal{M}}}, 1 \right\rangle \mathbf{I}, \\ &= n_i \tilde{\mathbf{I}}, \end{aligned} \quad (15)$$

where, again, ϵ is the volume form, \star is the Hodge dual, $\langle \cdot, \cdot \rangle$ is the inner product for forms (in this case 0-forms), \mathbf{I} is the unit (diagonal) form, and at the last step we have defined the density form $\tilde{\mathbf{I}} = \frac{1}{V_{\mathcal{M}}} \mathbf{I}$ (also diagonal). At this point, by the use of the De Rham's theorem we can think of the

characteristic densities to be elements of the $H^2(\mathcal{M}; \mathbb{R}) \times H^2(\mathcal{M}; \mathbb{R})$ kind, which are forced to be diagonal. If we interpret these to be diagonal intersection forms Q_M , by Donaldson's theorem we are ensuring that the considered manifold is smooth and simply connected [31]. Therefore, the topological numbers $n_i \in \mathbb{Z}$, i.e are known to be integers, necessarily.

The setup described above was considered in [10], where several particular cases were studied, provided that \mathcal{M} be *non-compact* in order to maintain the topological degrees of freedom. With this proviso, the variation of the action (12) yields the following field equations

$$\delta e : 0 = -\epsilon_{abcd}e^b \wedge R^{cd} + \frac{\Lambda_{eff}}{3}\epsilon_{abcd}e^b \wedge e^c \wedge e^d + d\varphi_{NY} \wedge T_a, \quad (16)$$

$$\delta \omega : 0 = -2\epsilon_{abcd}T^c \wedge e^d - 2d\varphi_P \wedge R_{ab} - d\varphi_{NY} \wedge e_a \wedge e_b - 2\epsilon_{abcd}d\varphi_E \wedge R^{cd}, \quad (17)$$

$$\delta \varphi : \frac{n_{NY}}{V_{\mathcal{M}}} \epsilon = T^a \wedge T_a - R_{ab} \wedge e^a \wedge e^b, \quad (18)$$

$$\delta \chi : \frac{n_E}{V_{\mathcal{M}}} \epsilon = \epsilon_{abcd}R^{ab} \wedge R^{cd}, \quad (19)$$

$$\delta \tau : \frac{n_P}{V_{\mathcal{M}}} \epsilon = R^a_b \wedge R^b_a. \quad (20)$$

With the additional condition that the couplings are some pullback of a common zero-form $\eta : \mathcal{M} \rightarrow \mathbb{R}$ such that $\varphi_i = \phi_i^* \eta$, it can be proven that the previous system of differential equations lead to the following expression for the torsion

$$T_a = -\left(\gamma \frac{\partial \varphi_P}{\partial \varphi_E} + \frac{\Lambda_{eff}}{3}\right) d\varphi_E \wedge e_a + \\ -\frac{1}{4\sigma} \left(\frac{2\Lambda_{eff}}{3} + \frac{\partial \varphi_{NY}}{\partial \varphi_P} + 8\sigma \gamma \frac{\partial \varphi_E}{\partial \varphi_P}\right) \epsilon_{abcd}e^b \wedge e^c \wedge \mathcal{L}^d(\varphi_P), \quad (21)$$

where $\sigma = +1$ when the metric is Euclidean-like or $\sigma = -1$ when Lorentzian-like, respectively, and $\mathcal{L}_a(\varphi_i) \doteq i_a(d\varphi_i)$ is the Lie derivative of zero forms and $i_{e_c}(\cdot) \equiv i_c(\cdot)$ is the interior (or slant) product with respect to the vierbein.

On the other hand, by a generalization of the procedure used in [10] the curvature two-form must be of the form

$$R_{ab} = \left(\frac{\Lambda_{eff}}{3}\right) e_a \wedge e_b + \sum_i d\varphi_i \wedge [\mathcal{L}_b(\varphi_i) e_a - \mathcal{L}_a(\varphi_i) e_b] + \\ + \sum_i \gamma_i d\varphi_i \wedge (e_a - e_b) + \gamma \epsilon_{abcd}e^c \wedge e^d, \quad (22)$$

where $\gamma : \mathcal{M} \rightarrow \mathbb{R}$ and $\gamma_i : \mathcal{M} \rightarrow \mathbb{R}$ are introduced as auxiliary zero-forms to be matched when comparing with equation (9). These auxiliary zero-forms fulfill the following restrictions

$$\begin{aligned} \frac{n_E}{4!V_{\mathcal{M}}} &= 2\gamma^2 + \left(\frac{\Lambda_{eff}}{3}\right)^2 - \left(\frac{\Lambda_{eff}}{3}\right) \mathcal{L}_a(\Sigma_i \varphi_i) \mathcal{L}^a(\Sigma_j \varphi_j) + \\ &\quad - \sum_j \gamma_j \left[\mathcal{L}_3(\varphi_j) + \mathcal{L}_0(\varphi_j) + \frac{2}{3}(\mathcal{L}_2(\varphi_j) + \mathcal{L}_1(\varphi_j)) \right], \end{aligned} \quad (23)$$

$$\frac{n_P}{4!V_{\mathcal{M}}} = -2\frac{\Lambda_{eff}}{3}\gamma. \quad (24)$$

It is important to mention that this generalization does not affect the solution of the torsion two form (equation (21)).

4. Getting General Relativity back: A Topological Cosmological Constant

Let us recall equation (14) for the effective cosmological constant, which we repeat here for later convenience. By setting $\Lambda = 0$, we get

$$\Lambda_{eff}^* \doteq \Lambda_{eff}|_{\Lambda=0} = \frac{1}{4V_{\mathcal{M}}} \sum_i n_i \varphi_i, \quad (25)$$

which only depends on the couplings φ_i and the topological numbers n_i . This means that, under suitable conditions, it is not necessary to include a cosmological constant *a priori* as the expression in (25) can provide the same effect. In other words, as it was suggested in [10], we can claim a *quasi topological origin* for the cosmological constant. We will prove that this *quasi topological* turns into *topological* if we impose those restrictions required by General Relativity.

Note that recovering GR is equivalent to set the conditions $T_a = 0$ and $R_{ab} \wedge e^b = 0$ that we pass on to study in a more detailed fashion on what follows.

4.1. Imposing $T_a = 0$.

Before embarking on this we need to recall that the previous solutions have been obtained by considering the couplings to be some pullback of a

common zero-form η on which we will heavily base our results. Note that when combining equations (16) and (17) we obtain the expression

$$T^b \wedge [2(\star\{e_b \wedge e_a\}) - \eta_{ab}(d\varphi_E \wedge d\varphi_{NY})] + d\varphi_P \wedge (d_\omega T_a) = \left(\frac{\Lambda_{eff}^*}{3}\right) d\varphi_E \wedge \epsilon_{abcd} e^b \wedge e^c \wedge e^d. \quad (26)$$

where η_{ab} is the Minkowski metric. The previous equation tell us that in the *absence of torsion* ($T_a = 0$) while at the same times having a *non zero effective cosmological constant* ($\Lambda_{eff}^* \neq 0$), we necessarily need to consider $d\varphi_E = 0$, or equivalently

$$\frac{\partial\varphi_E}{\partial\eta} = 0, \quad (27)$$

where η is the aforementioned common zero form. On the other hand, directly from (21), we get the following set of equations

$$0 = \gamma \frac{\partial\varphi_P}{\partial\varphi_E} + \frac{\Lambda_{eff}^*}{3}, \quad (28)$$

$$0 = \frac{2\Lambda_{eff}^*}{3} + \frac{\partial\varphi_{NY}}{\partial\varphi_P} + 8\sigma\gamma \frac{\partial\varphi_E}{\partial\varphi_P}. \quad (29)$$

These expressions link the Jacobians of the different coupling zero-forms φ_i . Finally, from equation (5), an immediate consequence of taking a null torsion is the fact that

$$n_{NY} = 0. \quad (30)$$

Hence, as it was expected, a topological restriction for the nullity of the Nieh-Yan number has appeared since it is the only topological class that solely depends on the torsion two-form.

4.2. Imposing $R_{ab} \wedge e^b = 0$.

Now we consider the second restriction. By imposing this on equation (22) we get the following system of equations

$$0 = \gamma, \quad (31)$$

$$0 = \sum_i \gamma_i d\varphi_i, \quad (32)$$

It is immediate now, from equations (28) and (24), that

$$n_P = 0. \quad (33)$$

Henceforth, a second strong restriction has been found for the Pontryagin number. Remember that for a four-manifold $n_E = p_1(T\mathcal{M}) = 0$, meaning that the space-time posses at least one orientation reversing diffeomorphism [22].

Note that at this point, by the aim of equations (27), (25), (30) and (33) we can write the effective cosmological constant Λ_{eff}^* as

$$\Lambda_{eff}^* = \left(\frac{\varphi_E^{(0)}}{4V_{\mathcal{M}}} \right) n_E, \quad (34)$$

where $\varphi_E^{(0)} \neq 0$ is a constant. Its character is still *quasi topological* due to the presence of $\varphi_E^{(0)}$. However, one immediate result is that, under the suggested conditions proposed, the *the Euler number must be different from zero*.

Note that equations (28), (29) and (31) are consistent only if we consider $\frac{\partial \varphi_P}{\partial \eta} \neq 0$, which gives the following equation between the remaining Jacobians

$$\frac{\partial \varphi_{NY}}{\partial \eta} = -2 \left(\frac{\Lambda_{eff}}{3} \right) \frac{\partial \varphi_P}{\partial \eta}. \quad (35)$$

Let us now consider equation (24), that by the aim of equations (35) and (29) can be written in the form

$$\frac{1}{2\varphi_E^{(0)}} = \left(\frac{\Lambda_{eff}}{3} \right) - \left[1 - 2 \left(\frac{\Lambda_{eff}}{3} \right) \right]^2 \left(\frac{\partial \varphi_P}{\partial \eta} \right)^2 [\mathcal{L}_a(\eta) \mathcal{L}^a(\eta)]. \quad (36)$$

The latter can be thought as a second order equation for $\frac{\Lambda_{eff}}{3}$ that generally will depend on the Lie derivatives of the auxiliary zero form η as well as the Jacobian $\frac{\partial \varphi_P}{\partial \eta} \neq 0$.

4.3. Eikonal propagation of topological information

At this point, the constancy of all the terms but one imposes it to be constant

$$\left(\frac{\partial \varphi_P}{\partial \eta} \right)^2 [\mathcal{L}_a(\eta) \mathcal{L}^a(\eta)] = [\mathcal{L}_a(\varphi_P) \mathcal{L}^a(\varphi_P)] = K, \quad (37)$$

which can be rewritten as

$$\mathcal{L}_a(\varphi_P) \mathcal{L}^a(\varphi_P) = \eta^{ab} \mathcal{L}_a(\varphi_P) \mathcal{L}_b(\varphi_P) = g^{\mu\nu} \partial_\mu \varphi_P \partial_\nu \varphi_P = K, \quad (38)$$

i.e. φ_P must satisfy the **inhomogeneous eikonal equation** in space time. This represents the propagation of Pontryagin topological information. As this result must be valid for any spacetime, the high energy regime imposes the eikonal equation to be homogeneous and thus $K = 0$ [32, 33].

4.4. the effective cosmological constant

Equation (38) inserted in (36) for $K = 0$ sets the value of $\varphi_E^{(0)}$ to

$$\varphi_E^{(0)} = \sqrt{\frac{6V_{\mathcal{M}}}{n_E}} \Rightarrow \frac{\Lambda_{eff}}{3} = \left(\frac{1}{4! V_{\mathcal{M}}}\right)^{\frac{1}{2}} (n_E)^{\frac{1}{2}}, \quad (39)$$

i.e. the $\Lambda_{eff}^* \propto (n_E)^{\frac{1}{2}}$. At this point it is important to mention the result by Berger [34, 35] that states that $n_E \geq 0$ for *compact manifolds*. This not a bound for us since we are considering *non-compact manifolds*. However, we are finding the stronger condition $n_E > 0$.

5. Conclusion

This work have discussed the application to the cosmological constant problem of the topological invariants examined in [10], introduced as Lagrange multipliers, that are compatible with GR for cosmological manifolds. At the level of the Einstein-Hilbert action, this results in the emergence of a topological effective cosmological constant from the Euler number. To do so, we have recovered GR by setting the resulting torsion to zero. This induces constraints on the topological invariants and yields the form of the topological effective cosmological constant, assuming that symmetry considerations have set the bare constant to zero. For torsion to be zero, both the Nieh-Yan and Pontryagin numbers have to be set to zero, leaving only the Euler number as source of effective cosmological constant.

On one hand this allows to propose a solution for the so-called fine tuning problem of the cosmological constant [20, 21], as the value observed is no longer related to the quantum vacuum energy density, but rather to the Euler number of the manifold, when it is of finite volume. In fact observations tend to favour slightly closed models [12, 13, 14, 15] so we expect that volume to be very large, hence favouring a small effective cosmological constant. On another hand, this opens the possibility of determining from observations that Euler number which describes some of the topological characteristics of the universe.

Recall that the Euler number can be measured locally by computing

$$\int_{\mathcal{M}} \epsilon_{abcd} R^{ab} \wedge R^{cd} = n_E,$$

that ultimately comes from the metric extrapolated from our measured local patch. However, in the case of the topological origin of the cosmological constant, as its value is unchanged over the whole manifold, any local measure reflects its global value.

It turns out that the smallness of the cosmological constant with respect to Planck density, one of the deepest puzzles of modern cosmology, is actually quite important considering the inverse volume of the universe and that consequently its value is very natural and leading to a positive Euler number determining the shape of the universe.

A dynamic theory of the topology of the universe in the line of emerging geometry [36, 37, 38, 39] should provide the framework for solving as well the coincidence problem [16, 17, 18, 19] by producing a mechanism of selection for the genus.

Acknowledgments

The work of M.Le D. has been supported by PNPd/CAPES20132029. M.Le D. also wishes to acknowledge IFT/UNESP.

References

- [1] P. Dona, S. Speziale, Introductory lectures to loop quantum gravity (2010).
- [2] D. Giulini, Ashtekar variables in classical general relativity (1993).
- [3] M. Nakahara, Geometry, Topology and Physics, Second Edition, Graduate student series in physics, Taylor & Francis, 2003.
- [4] A. Hatcher, Algebraic Topology, Cambridge University Press, 2002.
- [5] V. C. de Andrade, L. C. T. Guillen, J. G. Pereira, Teleparallel Gravity: An Overview, ArXiv General Relativity and Quantum Cosmology e-prints (2000).
- [6] J. C. Baez, D. K. Wise, Teleparallel Gravity as a Higher Gauge Theory, ArXiv e-prints (2012).
- [7] S. Sengupta, SU(2) gauge theory of gravity with topological invariants, J.Phys.Conf.Ser. 360 (2012) 012024.

- [8] R. K. Kaul, S. Sengupta, Topological parameters in gravity, *Phys.Rev. D* 85 (2012) 024026.
- [9] P. Baekler, F. W. Hehl, Beyond Einstein-Cartan gravity: Quadratic torsion and curvature invariants with even and odd parity including all boundary terms, *Class.Quant.Grav.* 28 (2011) 215017.
- [10] J. Lorca Espiro, Y. Vásquez, Einstein-Hilbert action nontrivially coupled with topological characteristic classes: generating Torsion and Contortion, *ArXiv e-prints* (2014).
- [11] S. Perlmutter, et al., Discovery of a supernova explosion at half the age of the Universe and its cosmological implications, *Nature* 391 (1998) 51–54.
- [12] G. Hinshaw, et al., Nine-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Parameter Results, *Astrophys.J.Suppl.* 208 (2013) 19.
- [13] C. Bennett, et al., Nine-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Final Maps and Results, *Astrophys.J.Suppl.* 208 (2013) 20.
- [14] N. Aghanim, et al., Planck intermediate results. III. The relation between galaxy cluster mass and Sunyaev-Zeldovich signal, *Astron.Astrophys.* 550 (2013) A129.
- [15] P. Ade, et al., Planck 2013 results. I. Overview of products and scientific results, *Astron.Astrophys.* 571 (2014) A1.
- [16] L. Amendola, Coupled quintessence, *Phys.Rev. D* 62 (2000) 043511.
- [17] D. Tocchini-Valentini, L. Amendola, Stationary dark energy with a baryon dominated era: Solving the coincidence problem with a linear coupling, *Phys.Rev. D* 65 (2002) 063508.
- [18] W. Zimdahl, D. Pavon, Interacting quintessence, *Phys.Lett. B* 521 (2001) 133–138.
- [19] W. Zimdahl, D. Pavon, Scaling cosmology, *Gen.Rel.Grav.* 35 (2003) 413–422.

- [20] S. Weinberg, The Cosmological Constant Problem, *Rev.Mod.Phys.* 61 (1989) 1–23.
- [21] S. Weinberg, *Cosmology* (2008).
- [22] S. Donaldson, P. Kronheimer, *The Geometry of Four-manifolds*, Oxford mathematical monographs, Clarendon Press, 1990.
- [23] T. Liko, Barbero-Immirzi parameter, manifold invariants and Euclidean path integrals, *Class.Quant.Grav.* 29 (2012) 095009.
- [24] Y. Cho, B. Park, D. Pak, A Minimal model of Lorentz gauge gravity with dynamical torsion, *Int.J.Mod.Phys. A*25 (2010) 2867–2882.
- [25] A. Hatzinikitas, Locally Weyl invariant massless bosonic and fermionic spin 1/2 action in the $(W(n)(4),g)$ and $(U(4),g)$ space-times, *Gen.Rel.Grav.* 32 (2000) 2287–2294.
- [26] M. Cambiaso, L. F. Urrutia, An extended solution space for Chern-Simons gravity: the slowly rotating Kerr black hole, *Phys.Rev. D*82 (2010) 101502.
- [27] R. G. Leigh, N. N. Hoang, A. C. Petkou, Torsion and the Gravity Dual of Parity Symmetry Breaking in $AdS(4)$ / $CFT(3)$ Holography, *JHEP* 0903 (2009) 033.
- [28] A. C. Petkou, Torsional degrees of freedom in AdS_4/CFT_3 (2010).
- [29] O. Chandia, J. Zanelli, Topological invariants, instantons and chiral anomaly on spaces with torsion, *Phys.Rev. D*55 (1997) 7580–7585.
- [30] O. Randal-Williams, Relations among tautological classes revisited, *ArXiv e-prints* (2010).
- [31] J. Milnor, J. Stasheff, *Characteristic Classes*, *Annals of mathematics studies*, Princeton University Press, 1974.
- [32] S. Frittelli, E. T. Newman, G. Silva-Ortigoza, The Eikonal equation in asymptotically flat space-times, *Journal of Mathematical Physics* 40 (1999) 1041–1056.

- [33] E. T. Newman, A. Perez, Characteristic surface data for the eikonal equation, *Journal of Mathematical Physics* 40 (1999) 1093–1102.
- [34] M. T. Anderson, A survey of Einstein metrics on 4-manifolds, *ArXiv e-prints* (2008).
- [35] C. LeBrun, Einstein Metrics, Four-Manifolds, and Differential Topology, *ArXiv Mathematics e-prints* (2004).
- [36] R. H. Brandenberger, String Gas Cosmology (2008) 193–230.
- [37] L. Perreault Levasseur, R. Brandenberger, A.-C. Davis, Defrosting in an Emergent Galileon Cosmology, *Phys.Rev. D* 84 (2011) 103512.
- [38] R. H. Brandenberger, Unconventional Cosmology, *Lect.Notes Phys.* 863 (2013) 333.
- [39] N. Afshordi, D. Stojkovic, Emergent Spacetime in Stochastically Evolving Dimensions, *Phys.Lett. B* 739 (2014) 117–124.